L12 Feb 3 Baire & New spaces

Tuesday, February 3, 2015 8:41 AM

Let $(X, \mathcal{T})$ be a space

A set $D \subset X$ is dense if $\overline{D} = X$

Qu. Recall the logical statement for it?

* $\forall x \in X, \underbrace{x \in \overline{D}}_{\forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset}$

* $\forall \emptyset \neq U \in \mathcal{T}, \quad U \cap A \neq \emptyset$

Examples

① $D = (0,1) \cup (1,2), \quad X = [0,2]$

② $D = \mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}, \quad X = \mathbb{R}$

③ Weierstrass Approximation

Any continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ can be approximated by polynomials on "cubes"

$\forall [a,b], \forall \varepsilon > 0 \ \exists$ polynomial $p : \mathbb{R}^n \to \mathbb{R}^m$ such that $\sup \{ \|\varphi(x) - p(x)\| : x \in [a,b]^n \} < \varepsilon$

$X = \{\text{continuous functions}\}$

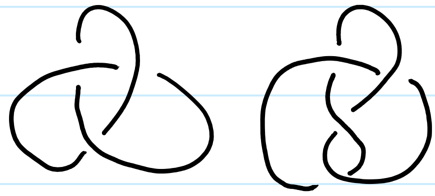
$\mathcal{T} =$ topology, called compact-open

$P = \{\text{polynomials}\}$

$\forall$ nbhd U of φ in X, $\exists\ p \in P$ with $p \in U$.

④ Knot Theory

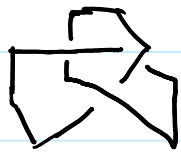
For any knot K in $\mathbb{R}^3$,



$\forall \varepsilon > 0$

$\exists$ polygonal knot L

$d(K, L) < \varepsilon$



Qu. What is the "opposite" of dense ?

A set $N \subset X$ is nowhere dense if $\mathrm{Int}(\overline{N}) = \emptyset$

Examples.

① $\mathbb{Z} \times \mathbb{Z}$ in standard $\mathbb{R}^2$

② $C = \left\{ (x(t), y(t)) \in \mathbb{R}^2 : t \mapsto (x(t), y(t)) \text{ differentiable} \right\}$

in standard $\mathbb{R}^2$

But, a continuous image may not be so!

e.g. space filling curve

Qu. What is the logical statement for $\mathrm{Int}(\overline{N}) = \emptyset$

$\forall\ x \in X \quad \underbrace{x \notin \mathrm{Int}(\overline{N})}$

$\forall\ U \in \mathcal{J}$ with $x \in U$, $\underbrace{U \not\subset \overline{N}}$

$U \cap (X \setminus \overline{N}) \neq \emptyset$

$\underbrace{\forall\ G \in \mathcal{J},\ G \setminus \overline{N} \neq \emptyset}$

$X \setminus \overline{N}$ is dense

L12 p3

Tuesday, February 3, 2015 9:04 AM

Qu. What is the topological difference of $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$?

Definition. $A \subset X$ is of first category, cat-I if

$$A = \bigcup_{k=1}^{\infty} N_k, \text{ each } N_k \text{ is nowhere dense}$$

Otherwise, it is of second category, cat-II

Facts

1. Clearly $\mathbb{Q}$ is of cat-I
2. Any countable set in $\mathbb{R}$ is of cat-I

   Any countable set in X is of cat-I ✗
3. A countable union of cat-I is cat-I.
4. Below, we will see $\mathbb{R}$ is of cat-II

   $\therefore$ $\mathbb{R}\setminus\mathbb{Q}$ is also of cat-II

Baire Category Theorem

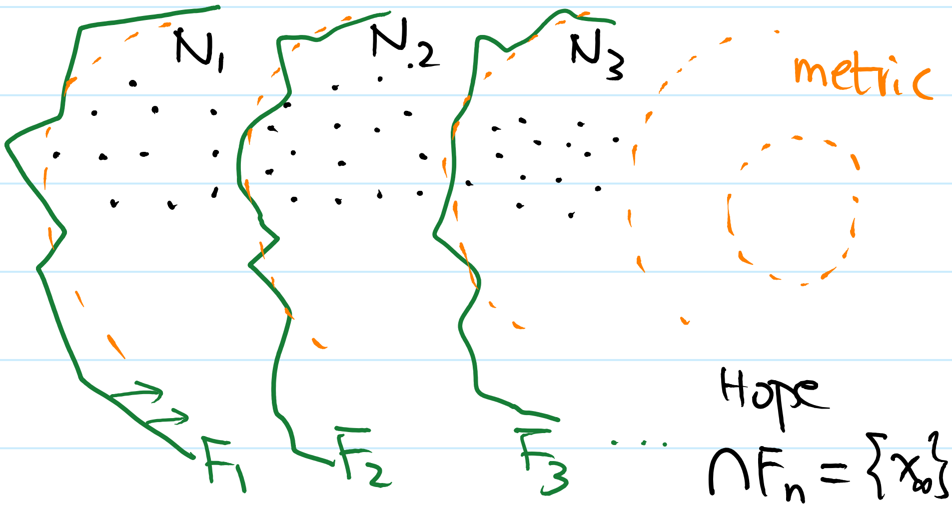
A complete metric space is always of cat-II.

Proof by contradiction, assume

$$X = \bigcup_{k=1}^{\infty} N_k, \text{ each } N_k \text{ nowhere dense}$$

Obviously, we will construct Cauchy Sequence

or indirectly — nested closed sets / contraction mapping — which?



Hope

$\cap F_n = \{x_\infty\} \not\subset \cup N_k$

$X \setminus \overline{N}_1 \neq \emptyset$

$\uparrow$ open, $\therefore \supset B(x_1, 2r_1) \supset F_1 = \{x \in X : d(x, x_1) \leq r_1\}$

$F_1 \supset B(x_1, r_1) \setminus (\overline{N}_1 \cup \overline{N}_2) \neq \emptyset$

$\uparrow$ open, $\therefore \supset B(x_2, 2r_2) \supset F_2$, $r_2 \leq \frac{r_1}{2}$

$F_2 = \{x \in X : d(x, x_2) \leq r_2\}$

So on, we have

$$F_1 \supset F_2 \supset F_3 \supset \cdots \cdots \supset F_n \supset F_{n+1} \supset \cdots$$

$$\operatorname{diam}(F_n) \leq \frac{r_1}{2^{n-1}} \longrightarrow 0$$

$$\{x_\infty\} = \bigcap_{n=1}^{\infty} F_n \subset X \setminus \left(\bigcup_{k=1}^{\infty} \overline{N}_k \right)$$

$$\subset X \setminus \bigcup_{k=1}^{\infty} N_k = \emptyset$$

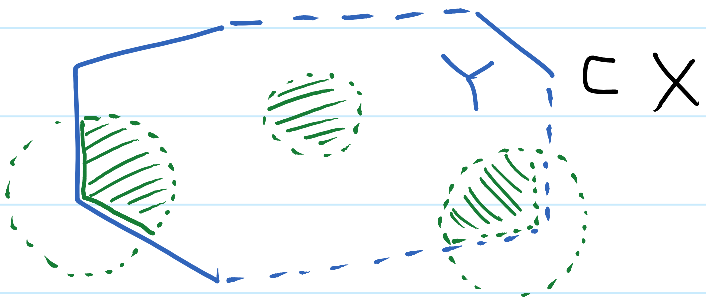
L12 New spaces

Tuesday, February 3, 2015 10:12 AM

Subspace Given $(X, \mathcal{T})$ and $\phi \neq Y \subset X$

$$\mathcal{T}|_Y = \{G \cap Y : G \in \mathcal{T}\}$$

Induced or Relative or Subspace Topology

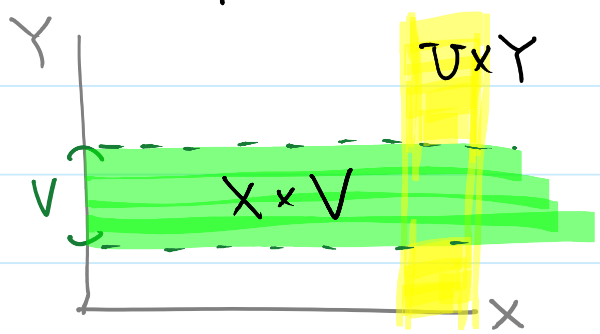


Finite Product Given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$

The product topology of $X \times Y$, $\mathcal{T}_{X \times Y}$

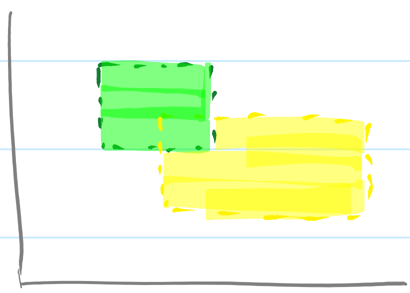
is generated by

$$\mathcal{S} = \{X \times V : V \in \mathcal{T}_Y\} \cup \{U \times Y : U \in \mathcal{T}_X\}$$



Their finite intersections give a base

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

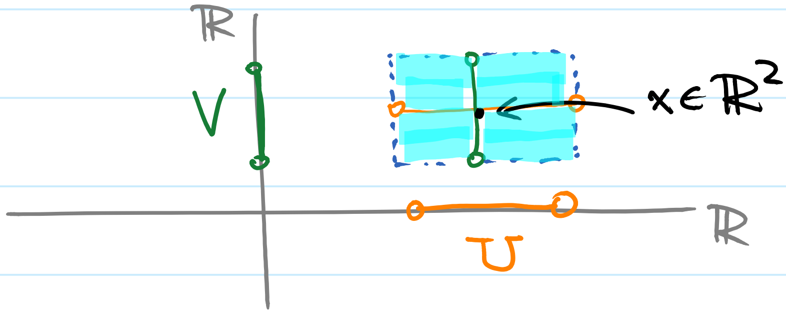


L12 p6

Tuesday, February 3, 2015 1:33 PM

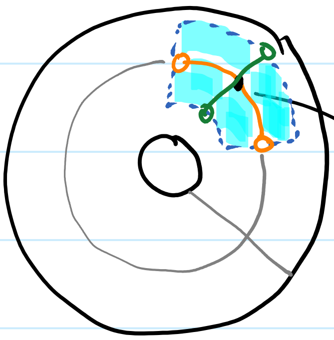
Intuition

① $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$



② $A = \{ z \in \mathbb{C} : 1 \leq |z| \leq 2 \}$ Annulus

as a subspace of $\mathbb{C} = \mathbb{R}^2$



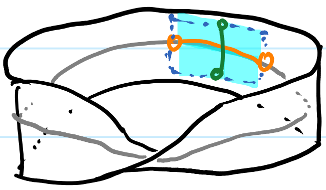
$z \in A \longrightarrow S^1 \times [0,1]$

product topology

$z \longmapsto \left(\frac{z}{|z|}, |z| - 1 \right)$

↑ $\arg z$

③ Warning



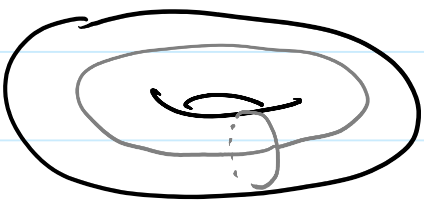
Every point has a nbhd of the form $U \times V$

$U \in \mathcal{T}_{S^1}$ $V \in \mathcal{T}_{[0,1]}$

But $M \neq S^1 \times [0,1]$

④ Torus

As a surface of revolution



$\subset \mathbb{R}^3 \longleftrightarrow S^1 \times S^1$

product space